Problem set solutions: MHD dynamos

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1 Tensor algebra

a) Compute the double contraction $\varepsilon_{ijk}\varepsilon_{ijl}$.

Solution: Using $\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$ leads too:
$\varepsilon_{ijk}\varepsilon_{ijl} = \delta_{jj}\delta_{kl} - \delta_{jl}\delta_{kj} = 3\delta_{kl} - \delta_{kl} = 2\delta_{kl}$.

b) Proof the vector identity

$$\boldsymbol{\nabla} \times (\boldsymbol{A} \times \boldsymbol{B}) = -(\boldsymbol{A} \cdot \boldsymbol{\nabla})\boldsymbol{B} + \boldsymbol{A}\boldsymbol{\nabla} \cdot \boldsymbol{B} + (\boldsymbol{B} \cdot \boldsymbol{\nabla})\boldsymbol{A} - \boldsymbol{B}\boldsymbol{\nabla} \cdot \boldsymbol{A}$$

Solution: Compute i^{th} component of expression:

$$\begin{aligned}
[\boldsymbol{\nabla} \times (\boldsymbol{A} \times \boldsymbol{B})]_i &= \varepsilon_{ijk} \frac{\partial}{\partial x_j} \left(\varepsilon_{klm} A_l B_m\right) \\
&= \varepsilon_{kij}\varepsilon_{klm} \left(\frac{\partial A_l}{\partial x_j} B_m + A_l \frac{\partial B_m}{\partial x_j} \right) \\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \left(\frac{\partial A_l}{\partial x_j} B_m + A_l \frac{\partial B_m}{\partial x_j} \right) \\
&= B_m \frac{\partial A_i}{\partial x_m} + A_i \frac{\partial B_m}{\partial x_m} - B_i \frac{\partial A_l}{\partial x_l} - A_l \frac{\partial B_i}{\partial x_l} \\
&= (\boldsymbol{B} \cdot \boldsymbol{\nabla}) A_i + A_i \boldsymbol{\nabla} \cdot \boldsymbol{B} - B_i \boldsymbol{\nabla} \cdot \boldsymbol{A} - (\boldsymbol{A} \cdot \boldsymbol{\nabla}) B_i
\end{aligned}$$

c) Let $a_{ij} = -\varepsilon_{ijk}\gamma_k$ be an antisymmetric tensor. Derive an expression for $\boldsymbol{\gamma}$.

Solution: Contract $a_{ij} = -\varepsilon_{ijk}\gamma_k$ with $-\frac{1}{2}\varepsilon_{lij}$:

$-\frac{1}{2}\varepsilon_{lij}a_{ij} = \frac{1}{2}\varepsilon_{lij}\varepsilon_{ijk}\gamma_k = \frac{1}{2}\underbrace{\varepsilon_{ijl}\varepsilon_{ijk}}_{2\delta_{lk}}\gamma_k = \gamma_l$

2 Second order correlation approximation

a) Start from the induction equation for $\boldsymbol{B}'$ (Volume I, Eq. 3.44):

$$\frac{\partial \boldsymbol{B}'}{\partial t} = \boldsymbol{\nabla} \times \left(\boldsymbol{v}' \times \overline{\boldsymbol{B}} + \overline{\boldsymbol{v}} \times \boldsymbol{B}' - \eta \boldsymbol{\nabla} \times \boldsymbol{B}' + \boldsymbol{v}' \times \boldsymbol{B}' - \overline{\boldsymbol{v}' \times \boldsymbol{B}'} \right) \ , \quad (1)$$

and assume $\overline{\boldsymbol{v}} = 0$, $|\boldsymbol{B}'| \ll |\overline{\boldsymbol{B}}|$ and neglect the contribution from magnetic resistivity. Formally integrate the equation to obtain a solution for $\boldsymbol{B}'$ and derive an expression for $\overline{\boldsymbol{\mathcal{E}}} = \overline{\boldsymbol{v}' \times \boldsymbol{B}'}$. Assume that $\boldsymbol{v}'$ has a finite correlation time, τ_c, and simplify expressions by approximating time integrals with $\int_{-\infty}^{t} \overline{v_i'(t) v_k'(s)} \mathrm{d}s = \tau_c \overline{v_i'(t) v_k'(t)}$.

Solution:
The simplified induction equation reads:

$$\frac{\partial \boldsymbol{B}'}{\partial t} = \boldsymbol{\nabla} \times \left(\boldsymbol{v}' \times \overline{\boldsymbol{B}} \right) \ ,$$

The formal solution for $\boldsymbol{B}'$ is:

$$\boldsymbol{B}'(t) = \int_{-\infty}^{t} \boldsymbol{\nabla} \times \left(\boldsymbol{v}'(s) \times \overline{\boldsymbol{B}(s)} \right) \mathrm{d}s \ .$$

The resulting emf reads:

$$\begin{aligned} \overline{\boldsymbol{\mathcal{E}}} &= \int_{-\infty}^{t} \overline{v'(t) \times \boldsymbol{\nabla} \times \left(\boldsymbol{v}'(s) \times \overline{\boldsymbol{B}}(s) \right)} \mathrm{d}s \approx \tau_c \overline{v' \times \boldsymbol{\nabla} \times \left(\boldsymbol{v}' \times \overline{\boldsymbol{B}} \right)} \\ &= \tau_c \overline{v' \times \left[(\overline{\boldsymbol{B}} \cdot \boldsymbol{\nabla}) \boldsymbol{v}' - \overline{\boldsymbol{B}} \boldsymbol{\nabla} \cdot \boldsymbol{v}' - (\boldsymbol{v}' \cdot \boldsymbol{\nabla}) \overline{\boldsymbol{B}} \right]} \end{aligned}$$

b) Express now all terms using the component notation summarized and show that the tensors a_{ij} and b_{ijk} in the expansion $\overline{\mathcal{E}}_i = a_{ij} \overline{B}_j + b_{ijk} \partial \overline{B}_j / \partial x_k$ are given by:

$$\begin{aligned} a_{ij} &= \tau_c \overline{\left(\varepsilon_{ikl} v_k' \frac{\partial v_l'}{\partial x_j} - \varepsilon_{ikj} v_k' \frac{\partial v_m'}{\partial x_m} \right)} && (2) \\ b_{ijk} &= \tau_c \varepsilon_{ijm} \overline{v_m' v_k'} \ . && (3) \end{aligned}$$

Solution:

$$\begin{aligned} \overline{\mathcal{E}}_i &= \tau_c \overline{\left\{ \varepsilon_{ikl} v_k' \left(\overline{B_j} \frac{\partial v_l'}{\partial x_j} - \overline{B_l} \frac{\partial v_m'}{\partial x_m} \right) - \varepsilon_{imj} v_m' v_k' \frac{\partial \overline{B}_j}{\partial x_k} \right\}} \\ &= \underbrace{\tau_c \overline{\left(\varepsilon_{ikl} v_k' \frac{\partial v_l'}{\partial x_j} - \varepsilon_{ikj} v_k' \frac{\partial v_m'}{\partial x_m} \right)}}_{a_{ij}} \overline{B_j} + \underbrace{\tau_c \varepsilon_{ijm} \overline{v_m' v_k'}}_{b_{ijk}} \frac{\partial \overline{B}_j}{\partial x_k} \end{aligned}$$

c) Decompose these tensors into the terms $\boldsymbol{\alpha}$, $\boldsymbol{\gamma}$ and $\boldsymbol{\beta}$ defined through:

$$\begin{aligned}\alpha_{ij} &= \frac{1}{2}\left(a_{ij}+a_{ji}\right)\\ \gamma_i &= -\frac{1}{2}\varepsilon_{ijk}a_{jk}\\ \beta_{ij} &= \frac{1}{4}\left(\varepsilon_{ikl}b_{jkl}+\varepsilon_{jkl}b_{ikl}\right)\ .\end{aligned}$$

Compute the trace α_{ii} and β_{ii}. To which physical quantities are they related?

Solution:

$$a_{ij} = \tau_c\overline{\left(\varepsilon_{ikl}v'_k\frac{\partial v'_l}{\partial x_j}-\varepsilon_{ikj}v'_k\frac{\partial v'_m}{\partial x_m}\right)}$$

Since the second term is antisymmetric in i and j it does not contribute to α_{ij}. Thus we have:

$$\begin{aligned}\alpha_{ij} &= \frac{1}{2}\tau_c\overline{\left(\varepsilon_{ikl}v'_k\frac{\partial v'_l}{\partial x_j}+\varepsilon_{jkl}v'_k\frac{\partial v'_l}{\partial x_i}\right)}\\ \alpha_{ii} &= \tau_c\varepsilon_{ikl}\overline{v'_k\frac{\partial v'_l}{\partial x_i}} = -\tau_c\overline{v'_k\varepsilon_{kil}\frac{\partial v'_l}{\partial x_i}} = -\tau_c\overline{\boldsymbol{v'}\cdot\boldsymbol{\nabla}\times\boldsymbol{v'}}\end{aligned}$$

$$\begin{aligned}\gamma_n = -\frac{1}{2}\varepsilon_{nij}a_{ij} &= -\frac{1}{2}\tau_c\overline{\left(\varepsilon_{nij}\varepsilon_{ikl}v'_k\frac{\partial v'_l}{\partial x_j}-\varepsilon_{nij}\varepsilon_{ikj}v'_k\frac{\partial v'_m}{\partial x_m}\right)}\\ &= -\frac{1}{2}\tau_c\overline{\left(\underbrace{\varepsilon_{ijn}\varepsilon_{ikl}}_{\delta_{jk}\delta_{nl}-\delta_{jl}\delta_{nk}}v'_k\frac{\partial v'_l}{\partial x_j}+\underbrace{\varepsilon_{nij}\varepsilon_{kij}}_{2\delta_{kn}}v'_k\frac{\partial v'_m}{\partial x_m}\right)}\\ &= -\frac{1}{2}\tau_c\overline{\left(v'_k\frac{\partial v'_n}{\partial x_k}-v'_n\frac{\partial v'_j}{\partial x_j}+2v'_n\frac{\partial v'_j}{\partial x_j}\right)}\\ &= -\frac{1}{2}\tau_c\frac{\partial}{\partial x_m}\overline{v'_nv'_m}\end{aligned}$$

With $b_{ijk}=\tau_c\varepsilon_{ijm}\overline{v'_mv'_k}$ we get:

$$\begin{aligned}\beta_{ij} &= \frac{1}{4}\left(\varepsilon_{ikl}b_{jkl}+\varepsilon_{jkl}b_{ikl}\right) = \frac{1}{4}\tau_c\left(\varepsilon_{ikl}\varepsilon_{jkm}+\varepsilon_{jkl}\varepsilon_{ikm}\right)\overline{v'_mv'_l}\\ &= \frac{1}{2}\tau_c\varepsilon_{ikl}\varepsilon_{jkm}\overline{v'_mv'_l} = \frac{1}{2}\tau_c\left(\delta_{ij}\delta_{lm}-\delta_{im}\delta_{jl}\right)\overline{v'_mv'_l}\\ &= \frac{1}{2}\tau_c\left(\delta_{ij}\overline{v'^2}-\overline{v'_iv'_j}\right)\\ \beta_{ii} &= \tau_c\overline{v'^2}\end{aligned}$$

α_{ii} is proportional to the negative kinetic helicity of the flow, β_{ii} is proportional to the turbulent rms velocity squared. γ can be expressed as the divergence of the velocity correlation tensor.

d) Make now the additional assumption of isotropy, which implies that α_{ij}, β_{ij}, as well as the correlation tensor $\overline{v'_i v'_j}$ are diagonal, i.e. $\alpha_{ij} = \alpha\delta_{ij}$. Compute the scalar α-effect and the turbulent diffusivity η_t. How is $\boldsymbol{\gamma}$ related to η_t? Discuss under which conditions these effects exist.

Solution:
Isotropy implies:

$$\begin{aligned}
\alpha &= \frac{1}{3}\alpha_{ii} = -\frac{1}{3}\tau_c\overline{\boldsymbol{v}' \cdot \boldsymbol{\nabla} \times \boldsymbol{v}'} \\
\eta_t &= \frac{1}{3}\beta_{ii} = \frac{1}{3}\tau_c\overline{v'^2} \\
\gamma_i &= -\frac{1}{2}\tau_c\frac{\partial}{\partial x_m}\overline{v'_i v'_m} = -\frac{1}{2}\tau_c\frac{\partial}{\partial x_m}\left(\frac{1}{3}\overline{v'^2}\delta_{im}\right) \\
&= -\frac{1}{6}\tau_c\frac{\partial}{\partial x_i}v'^2 = -\frac{1}{2}\frac{\partial}{\partial x_i}\eta_t
\end{aligned}$$

Note that the last step is only valid if τ_c does not vary spatially. Although this effect is very often expressed as gradient of η_t, this is not the case for highly stratified convection such as the solar convection zone. Since v'^2 is increasing monotonically from the base of the CZ toward the photosphere, the resulting γ describes a downward transport throughout the entire CZ "turbulent pumping".

η_t is present under minimal assumptions (e.g. isotropy, homogeniety) since it is simply related to the turbulence intensity. γ requires in addition inhomogeneity (e.g. stratification). For α reflectional symmetry needs to be broken, e.g. through a combination of stratification and rotation.